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The rotation of the Earth's inner core

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The inner core rotates under the influence of inertial, frictional, electromagnetic and gravity torques. When its symmetry axis is inclined by an angle θ_i to the mantle symmetry axis, calculation shows the gravity restoring torque to be $2.50 \times 10^{24} \cos\theta_i \sin\theta_i$ N m, several orders of magnitude larger than the other torques combined. Investigation of its rotational dynamics under this torque shows that it can have two modes of steady precession. One mode involves a slow precession in space and is presumably ruled out because of the nearly diurnal relative motion required with respect to the outer core. The second steady precessional mode involves a rapid, nearly diurnal prograde motion in space and hence is slow in the mantle frame. Because the inner core is a good electrical conductor and because of its large rotational energy it is likely to have a strong interaction with the main magnetic field. In particular, if it is in the second steady precessional mode it may be carrying magnetic fields bodily for periods of several thousand years (its magnetic diffusion time) and the 'dipole wobble' of the main field, seen in historical measurements, and apparently extending into the palaeomagnetic record, may be a reflection of the influence of this motion on the outer core dynamo.

The Earth is known to possess an inner core of roughly 1230 km radius which was discovered through its reflections of P-waves by Inge Lehmann in 1936. It was shown by Bullen in 1946 to be solid on the basis of sharply increased P-velocity there. With a possibly smaller fraction of light elements (Anderson 1983) it is expected to have a somewhat higher electrical conductivity than the value of 3×10^5 S m^{-1} usually assigned to the outer liquid core. Integration of the Clairaut equation shows it to have a flattening or ellipticity of 1 part in 416.

Apart from brief considerations by Braginsky (1964), Steenbeck & Helmis (1975) and by Gubbins (1981), the role of the inner core, if any, in the processes generating the geomagnetic field and its rotational dynamics has not been the subject of detailed analysis. Here we suggest that there is some geomagnetic evidence that it may play an active dynamical part in the time evolution of the main magnetic field.

Illustrated in figure 1 are successive positions of the pole of the dipole part of the magnetic field from 1600 to 1962.5 after the analyses of Barraclough (1974). The motion in longitude amounts to $0.08 \pm 0.01°$ per year while in latitude it is only $0.01 \pm 0.001°$ per year. An earlier analysis by Nagata (1965) gave 0.05° per year as the precessional rate of the dipole part of the geomagnetic field.

A clear distinction between dipole precession and the time evolution of the non-dipole field

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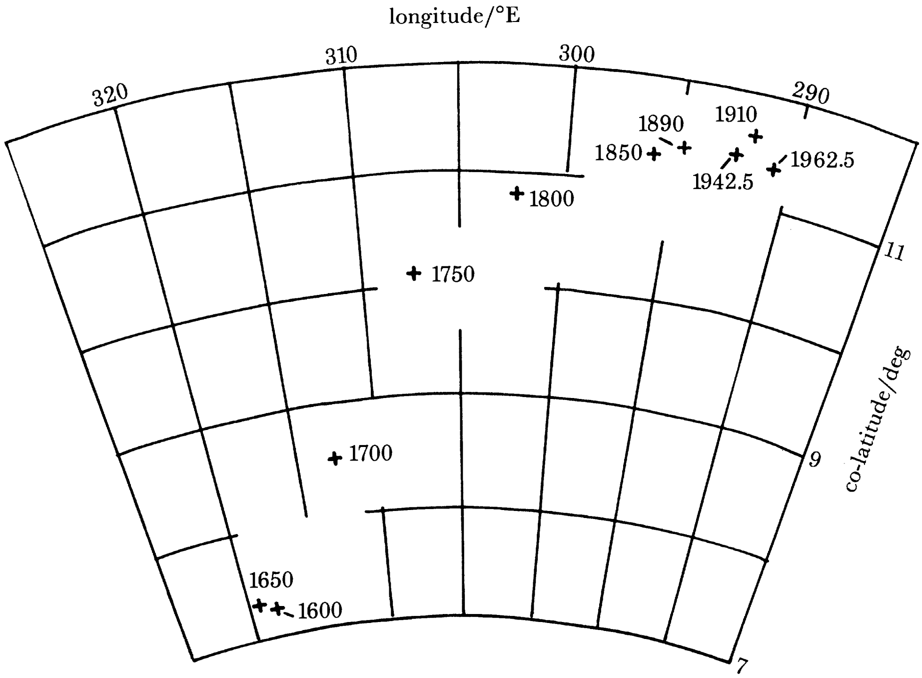

FIGURE 1. Movement of the pole of the dipole field from 1600 to 1962.5 after Barraclough (1974).

is revealed by the Bauer plot in figure 2 of inclination against declination at London, taken from the work of Malin & Bullard (1981). The dots represent values at five year intervals from 1575 to 1975. The solid curve represents the trajectory that would be followed for a uniform precession of a dipole field inclined at its present value of approximately 11° to the geographic axis. An arc corresponding to a dipole precession of 135° is shown and at a rate of 0.08° per year it would take nearly 1700 years to traverse, more than four times slower than the observed motion. Similar results are found for the rate of westward drift of the non-dipole field compared with that for dipole precession (Nagata 1965).

Another way of illustrating the slower time evolution of the dipole part of the field in contrast to that of the non-dipole field is to compute 'effective periods' for each field harmonic degree. These can be constructed by taking the mean square of the field magnitude corresponding to a given degree harmonic, and dividing it by the same quantity for the secular variation rate. The square root of this ratio provides a typical reciprocal angular velocity, allowing an 'effective period' to be calculated. Results are shown in table 1 for the field in 1965, calculated from the spherical harmonic decomposition due to Barraclough *et al.* (1978). If T_n is the 'effective period' for harmonic degree n then:

$$T_n = 2\pi(\overline{|\boldsymbol{B}_n|^2}/\overline{|\dot{\boldsymbol{B}}_n|^2})^{\frac{1}{2}}, \quad (1)$$

where $\overline{|\boldsymbol{B}_n|^2}$ is the mean square of the field of degree n averaged over the Earth's surface and $\overline{|\dot{\boldsymbol{B}}_n|^2}$ is the same quantity for the secular variation field. Comparable results are found for field models at other epochs (McQueen *et al.* 1983). The time scale for changes in the dipole part of the geomagnetic field is seen to exceed those for changes in the non-dipole part by about an order of magnitude as indicated by the 'effective periods' computed in this fashion.

There is also some evidence that 'dipole wobble' exists in the palaeomagnetic record as well

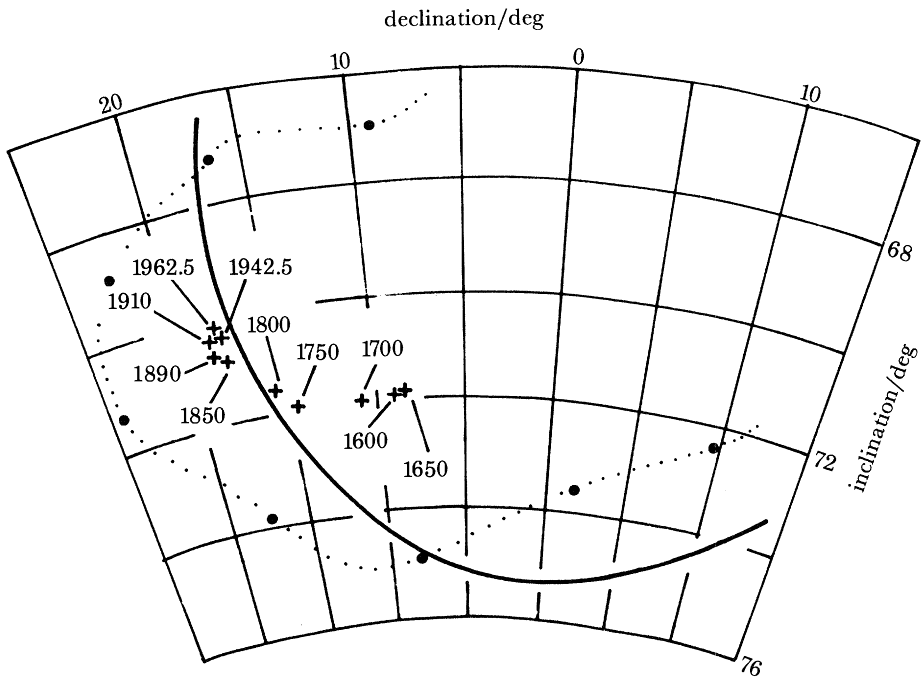

FIGURE 2. Bauer plot for the magnetic field at London from 1575 to 1975. The dots represent a plot of inclination against declination at five year intervals after Malin & Bullard (1981). Trajectory for precession of dipole field inclined at 11° to rotation axis is shown as a solid curve. The crosses represent successive dipole positions as computed by Barraclough (1974).

TABLE 1. 'EFFECTIVE PERIODS' T_n IN YEARS AS A FUNCTION OF HARMONIC DEGREE n

(Calculated for the 1965 field from the spherical harmonic decomposition of Barraclough *et al.* (1978).)

harmonic degree, n	'effective period', T_n
1	8367
2	866
3	1219
4	1279
5	648
6	421
7	334
8	172

as in historical magnetic variation measurements. Figure 3 shows the dependence of the angular dispersion of virtual geomagnetic poles on latitude for the Brunhes magnetic epoch over the last 700000 years. The plot is reproduced from the paper of McElhinney & Merrill (1975) and is based on data collected by Doell & Cox (1972). The solid curve indicates the theoretical dispersion to be expected if 'dipole wobble' or precession with the present day 11° inclination to the geographic axis were to have persisted in the past. McElhinney & Merrill (1975) conclude that 'dipole wobble' may extend back at least 5 Ma with an average inclination of about 9°.

The persistence of 'dipole wobble' for 10^5 to possibly more than 10^6 years suggests that it is a much more ordered phenomenon than the complex secular variation of the non-dipole field. Together with its much longer timescale for variation and its significantly slower precessional rate compared with the non-dipole part, this behaviour of the dipole field indicates that it is

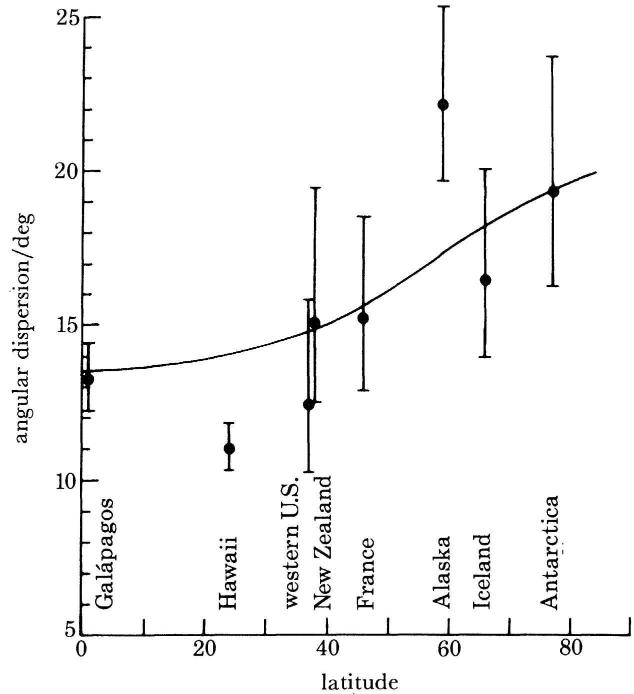

FIGURE 3. Angular dispersion of virtual geomagnetic poles as a function of latitude for Brunhes epoch after Doell & Cox (1972). Curve shows expected theoretical dispersion for a 'dipole wobble' of 11°.

likely that a distinct physical mechanism influences its time evolution. Our suggestion is that this mechanism may involve a bodily precession of the solid inner core.

Because of the presumed high electrical conductivity of the inner core, magnetic fields will be slow to diffuse in and out of it. On timescales short compared with the diffusion time, fields will be carried with the motion of the inner core, while on timescales long compared with the diffusion times, the inner core can be expected to acquire the long term ambient field of the outer core.

TABLE 2. MAGNETIC DIFFUSION TIMES τ_i FOR POLOIDAL AND TOROIDAL FIELDS IN RIGID SPHERES OF RADIUS a AND UNIFORM CONDUCTIVITY $\sigma = 3 \times 10^5$ S m^{-1}.

secular equation	poloidal field/deg	toroidal field/deg	longest diffusion time/years a = 1229.5 km	longest diffusion time/years a = 3484.3 km
$\sin \alpha_i a = 0$	1	—	1830	14695
$\tan \alpha_i a = \alpha_i a$	2	1	894	7183
$\tan \alpha_i a = 3\alpha_i a/(3+\alpha_i^2 a^2)$	3	2	544	4366
⋮	⋮	⋮	⋮	⋮
$\tau_i = \mu_0 \sigma/\alpha_i^2$				

The magnetic diffusion times of the inner core cannot be calculated exactly because they depend on the behaviour of the surrounding medium, the conducting fluid outer core. Table 2 gives the results for two extreme models. In the first, the inner core is taken to be surrounded by an insulator. Then the diffusion times become those for a sphere of radius a equal to the mean radius of the inner core (1229.5 km). In the second, it is taken to be surrounded by a rigid outer core with the same electrical conductivity. The diffusion times in this case are those for a sphere of radius a equal to the mean radius of the outer core (3484.3 km). In both cases we have taken the electrical conductivity to be 3×10^5 S m^{-1}. Of course, the diffusion of field harmonic of a given degree is governed by an infinite sequence of diffusion times. For simplicity, we have shown only the longest dominant member of each sequence.

The actual timescales for magnetic field changes in the inner core can be regarded as being bounded by the extreme values given in table 2. On one hand, the presence of the conducting liquid outer core will inhibit the diffusion so that timescales will be longer than those calculated for an inner core surrounded by an insulator. On the other, there are a variety of hydromagnetic mechanisms that will shorten the diffusion timescales compared with those calculated assuming the outer core to be a rigid conductor. Dipole precession rates of 0.05° to 0.08° per year yield precession periods in the range 4500–7200 years, consistent with the timescales in table 2 over which dipole magnetic field might be carried bodily by the conducting inner core.

The question that then arises is whether or not the inner core is dynamically capable of a slow precessional motion in the mantle frame. To answer this question it is necessary to consider the torques acting on the inner core. Because of its presumed high conductivity and the presence of strong magnetic fields in the outer core, the inner core is subject to electromagnetic torques (Gubbins 1981). Motion relative to the fluid outer core will be resisted by viscous drag, most probably due to turbulent eddy diffusion, since the molecular viscosity of the outer core is small (Toomre 1966; Rochester 1970). When the motion involves inclination of the inner core axis to the mantle axis, the slight ellipticity of the inner core produces an inertial restoring torque (Busse 1970; Kakuta *et al.* 1975). The gravity-restoring torque in such motions, however, is nearly five orders of magnitude larger than the inertial restoring torque because of the fact that the internal level surfaces in the Earth have flattenings that vary with radius in accordance with the Clairaut equation (Smylie *et al.* 1984). Because the gravity couple is along the same axis, the line of intersection of the equatorial planes of the inner core and mantle, the inertial restoring torque is completely negligible in the rotational dynamics of the inner core.

The inner core has a rotational inertia which is less than 10^{-3} that of the rest of the Earth. A useful first approach to its rotational dynamics, therefore, is to neglect the small compensating motion of the mantle and to consider its behaviour under the dominant gravity couple alone. Initially, then, the mantle is taken to be rotating uniformly at angular velocity $\boldsymbol{\Omega}$ along a fixed spatial direction indicated by the unit vector $\hat{\boldsymbol{e}}_3$. Successive rotations through the Euler angles ϕ_i, θ_i and Ψ_i carry the mantle frame with unit vectors $(\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3)$ into the inner core frame with unit vectors $(\hat{\boldsymbol{\varepsilon}}_1, \hat{\boldsymbol{\varepsilon}}_2, \hat{\boldsymbol{\varepsilon}}_3)$ as illustrated in figure 4.

Details of the calculation of the gravity-restoring torque have been presented previously (Szeto & Smylie 1984; Smylie *et al.* 1984). It is found to be reducible to the simple form:

$$\boldsymbol{\Gamma} = -\hat{\boldsymbol{\varepsilon}}_1' \tfrac{2}{5}(C_i - A_i)\{(GM_i/r_0^3)(2f_i + r_0 f_i') + (GM_i'/r_0^3)(3f_i)\}\cos\theta_i \sin\theta_i. \tag{2}$$

Here r_0 is the mean radius of the inner core and the difference $C_i - A_i$ between its axial and equatorial moments of inertia is given by the usual integral of mass density ρ_0 and flattening f over radius.

$$C_i - A_i = \tfrac{8}{15}\pi \int_0^{r_0} \rho_0 \frac{d}{dr_0}(r_0^5 f)\, dr_0. \tag{3}$$

In the torque expression (2), M_i is the mass of the inner core, M_i' is the mass it would have if it had the same density ρ_i as the material just outside its boundary, and f_i, f_i' are respectively the flattening and the radial derivative of the flattening at the surface of the inner core. The

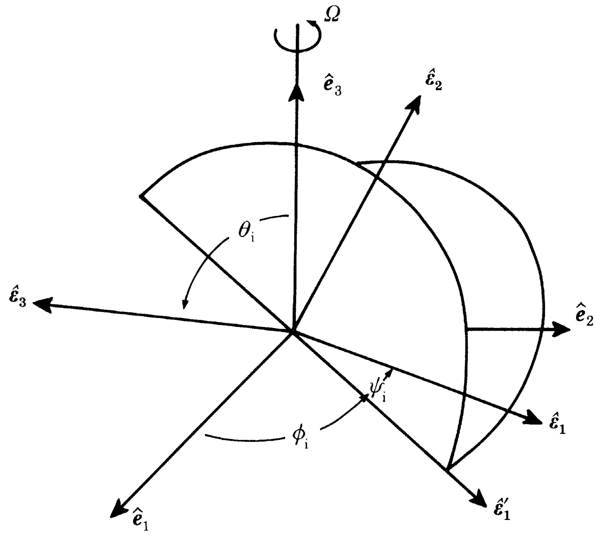

FIGURE 4. Transformation from mantle frame $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ to inner core frame $(\hat{\varepsilon}_1, \hat{\varepsilon}_2, \hat{\varepsilon}_3)$. Mantle to inner core transformation is $\phi_i \hat{e}_3 \to \theta_i \hat{\varepsilon}_1' \to \psi_i \hat{\varepsilon}_3$.

TABLE 3. RESULTS OF INTEGRATING THE CLAIRAUT EQUATION IN THE INNER CORE FOR EARTH MODEL 1066A OF GILBERT & DZIEWONSKI (1975)

(Integration was done with $m = \Omega^2 d^3/GM = 3.449805 \times 10^{-3}$ adopted by the I.A.G. (1971). f' was determined by cubic spline interpolation and $C_i - A_i$ was calculated by using third-order accurate integration. For earth model 1066A, $\rho_i = 12.153 \times 10^3$ kg m^{-3}, giving $M_i' = \frac{4}{3}\pi r_0^3 \rho_i = 9.462 \times 10^{22}$ kg. I_i is the mean moment of inertia of the inner core.)

$10^{-3} r_0$/m	$10^{-3} \rho_0$/kg m^{-3}	$1/f$	$10^{11} f'$/m^{-1}	$10^{-30}(C_i - A_i)$ /kg m^2	$10^{-21} M_i$/kg	$10^{-32} I_i$/kg m
0	13.421	418.67	0.473	—	0	—
115.2	13.406	418.56	0.627	—	0.09	—
230.4	13.385	418.42	0.774	0.04	0.69	0.15
345.6	13.357	418.25	0.963	0.27	2.31	1.13
460.8	13.316	418.03	1.243	1.11	5.47	4.66
576.0	13.267	417.75	1.525	3.39	10.66	14.15
691.3	13.220	417.42	1.693	8.41	18.39	35.09
806.5	13.175	417.07	1.752	18.13	29.12	75.63
921.7	13.131	416.72	1.764	35.28	43.36	147.02
1036.9	13.088	416.37	1.763	63.45	61.58	264.18
1152.1	13.045	416.02	1.754	107.24	84.26	446.12
1229.5	13.021	415.78	1.743	148.25	102.24	616.39

foregoing quantities are listed in table 3 for the Earth model 1066A of Gilbert & Dziewonski (1975).

When numerical values are inserted in relation (2) we obtain

$$\boldsymbol{\Gamma} = -\hat{\varepsilon}_1' \Gamma \cos\theta_i \sin\theta_i, \tag{4}$$

with $\Gamma = 2.50 \times 10^{24}$ N m.

The angular velocity of the inner core with respect to the mantle frame is:

$$\boldsymbol{\omega}_i = \dot{\phi}_i \hat{e}_3 + \dot{\theta}_i \hat{\varepsilon}_1' + \dot{\psi}_i \hat{\varepsilon}_3. \tag{5}$$

Its spatial angular velocity is then

$$\boldsymbol{\omega} = \boldsymbol{\omega}_i + \Omega \hat{\boldsymbol{e}}_3 = (\dot{\theta}_i \cos\phi_i + \dot{\psi}_i \sin\phi_i \sin\theta_i)\,\hat{\boldsymbol{e}}_1 + (\dot{\theta}_i \sin\phi_i - \dot{\psi}_i \cos\phi_i \sin\theta_i)\,\hat{\boldsymbol{e}}_2 + (\Omega + \dot{\phi}_i + \dot{\psi}_i \cos\theta_i)\,\hat{\boldsymbol{e}}_3, \quad (6)$$

expressed in the mantle frame. In the inner core frame it is:

$$\boldsymbol{\omega} = (\Omega \sin\theta_i \sin\psi_i + \dot{\phi}_i \sin\theta_i \sin\psi_i + \dot{\theta}_i \cos\psi_i)\,\hat{\boldsymbol{\epsilon}}_1 + (\Omega \sin\theta_i \cos\psi_i + \dot{\phi}_i \sin\theta_i \cos\psi_i - \dot{\theta}_i \sin\psi_i)\,\hat{\boldsymbol{\epsilon}}_2 + (\Omega \cos\theta_i + \dot{\psi}_i + \dot{\phi}_i \cos\theta_i)\,\hat{\boldsymbol{\epsilon}}_3. \quad (7)$$

The direction cosines of the inner core axis in the mantle frame are $\gamma_1 = \sin\phi_i \sin\theta_i$, $\gamma_2 = -\cos\phi_i \sin\theta_i$ and $\gamma_3 = \cos\theta_i$. Thus, in the mantle frame the inner core inertia tensor becomes

$$A_i I + (C_i - A_i) \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix} (\gamma_1, \gamma_2, \gamma_3), \quad (8)$$

where I is the identity matrix.

The gravity-restoring torque has no component along either the mantle axis or the inner core axis. While the latter axis is moving in space, because it is an axis of symmetry, no correction is required to the time derivative of the component of angular momentum along it. The angular momentum is then constant along both axes and two first integrals of the equations of motion are obtained immediately. They may be solved for the angular velocities $\dot{\phi}_i$, $\dot{\psi}_i$ to give

$$\dot{\phi}_i = -\Omega + \frac{(b - a\cos\theta_i)}{\sin^2\theta_i}, \quad \dot{\psi}_i = \Omega_i - \frac{(b - a\cos\theta_i)}{\sin^2\theta_i}\cos\theta_i. \quad (9)$$

The spatial spin rate of the inner core Ω_i is a constant of the motion as are a and b; a is simply an abbreviation for $C_i \Omega_i / A_i$ which gives the constant three-component of the product of the inertia tensor in the inner core frame with the angular velocity (7) as $A_i a$. Similarly, the constant three-component of the product of the inertia tensor (8) with the angular velocity (6) is $A_i b$, defining the quantity b which also has the dimensions of an angular velocity.

Since our model of the rotational dynamics of the inner core is conservative, a third constant of the motion is the total energy E. The kinetic energy is

$$T = \tfrac{1}{2} A_i \{(\dot{\phi}_i + \Omega)^2 \sin^2\theta_i + \dot{\theta}_i^2\} + \tfrac{1}{2} C_i \Omega_i^2, \quad (10)$$

while integration of the torque expression (4) yields

$$V = \tfrac{1}{2}\Gamma \sin^2\theta_i \quad (11)$$

for the potential energy. By using the quantity $E' = E - \frac{1}{2} C_i \Omega_i^2$ to measure the excess of the total energy over the kinetic energy associated with constant rotation of the inner core about its symmetry axis, with the variable transformation $u = \cos\theta_i$, the total energy equation can be written

$$f(u) = (1-u^2)\,\dot{\theta}_i^2 = \dot{u}^2 = (1-u^2)\{\alpha - \beta(1-u^2)\} - (b - au)^2, \quad (12)$$

where the first of equations (9) has been used to eliminate $\dot{\phi}_i$ from the kinetic energy expression (10) and where $\alpha = 2E'/A_i$, $\beta = \Gamma/A_i$. The function $f(u)$ is the quartic in u shown plotted schematically in figure 5. In general, the kinetic energy will exceed that of simple rotation about

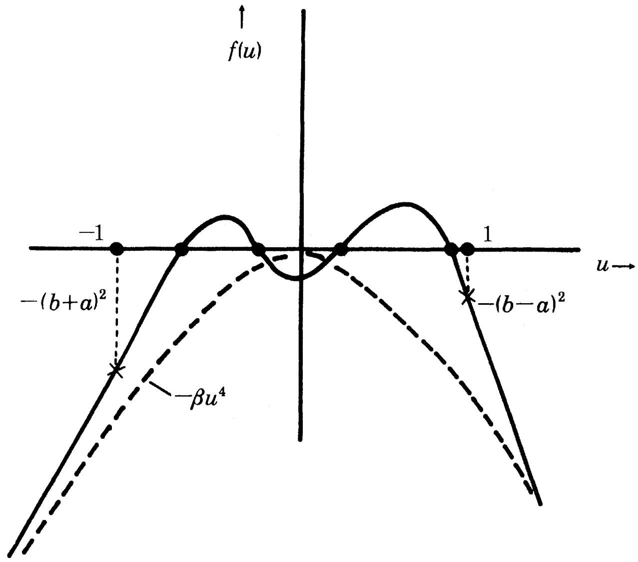

FIGURE 5. Energy quartic schematically illustrated showing four roots that determine the turning points of the inner core axis.

the inner core axis at the rate Ω_i by the amount $\Delta T = T - \frac{1}{2}C_i \Omega_i^2$ and the potential energy V will be positive so that $E' > 0$. Therefore, α and β will in general be positive and

$$\alpha - \beta(1-u^2) = (2/A_i)(E'-V) = 2\Delta T/A_i \geqslant 0. \tag{13}$$

Because $(b-au)^2$ is in general positive, the energy quartic (12) can only have roots for $u^2 < 1$. Thus, all of its real roots must lie on the physical portion of the line $|u| < 1$.

Since the roots of the energy quartic represent the values of $u = \cos\theta_i$ at which $\dot{\theta}_i$ vanishes they are seen to determine the turning angles which limit the motion of the inner core axis. In the most general situation there will be one pair of turning angles at smaller co-latitude and one pair at larger co-latitude bounding the nutational motion of the precessing inner core. Of course, except in periods of disturbance, such as during a magnetic reversal, we would expect these nutational motions to be damped out and the precession of the core to be regular. In regular precession $f(u)$ must have a double root and the condition for this is easily found to be

$$(\dot{\phi}_i + \Omega)^2 \cos\theta_i - (\dot{\theta}_i + \Omega)\, a - \beta \cos\theta_i = 0, \tag{14}$$

a quadratic equation in $\dot{\phi}_i + \Omega$ with the two roots

$$\dot{\phi}_i + \Omega = \frac{C_i \Omega_i}{2A_i \cos\theta_i}\left(1 \pm \left[1 + \frac{4A_i \Gamma \cos^2\theta_i}{C_i^2 \Omega_i^2}\right]^{\frac{1}{2}}\right). \tag{15}$$

Rapid motions of the inner core relative to the outer core and mantle are presumably inadmissible since they would be vigorously opposed by frictional and electromagnetic torques. Thus, the numerical value of Ω_i must be close to $\Omega\cos\theta_i$. Hence, the second term in the root in expression (15) is very nearly equal to

$$4\Gamma/I_i\,\Omega^2 = 3.05 \times 10^{-2}. \tag{16}$$

To accuracy of parts in 10^4, the rates of the two modes of regular precession are then given by

$$\dot{\phi}_i + \Omega = -\Gamma \cos\theta_i / C_i \Omega_i \tag{17}$$

and

$$\dot{\phi}_i = \left(\frac{\Omega_i}{\cos\theta_i} - \Omega\right) + \frac{1}{\cos\theta_i}\left(\sigma_i + \frac{\Gamma\cos^2\theta_i}{C_i\Omega_i}\right), \tag{18}$$

where $\sigma_i = [(C_i - A_i)/A_i]\,\Omega_i$ is the Euler angular frequency of the inner core.

To about one part in $C_i \Omega^2/\Gamma = 131$, the rate given by (17) for the first mode is $\dot{\phi}_i = -\Omega$. It is therefore a 'slow precession' in space, retrograde for $\theta_i < \frac{1}{2}\pi$ and prograde for $\theta_i > \frac{1}{2}\pi$. In contrast, the two terms in brackets on the right side of (18) are both small compared with Ω and hence the second mode is a 'rapid precession' in space, which is closely diurnal and prograde, making it slow in the mantle frame. It is this mode which is of geophysical interest since it permits the slow precession required to associate the time evolution of the dipole magnetic field with inner core bodily motion.

In both modes of regular precession, θ_i is by definition constant. Expressions (17) and (18) then imply that $\dot{\phi}_i$ is constant in both. Since the three-component of the right side of relation (7) yields

$$\Omega_i = (\dot{\phi}_i + \Omega)\cos\theta_i + \dot{\psi}_i, \tag{19}$$

$\dot{\psi}_i$ is also constant in both regular precessional modes. In the 'slow precession' mode it is

$$\dot{\psi}_i = \Omega_i + \Gamma\cos^2\theta_i / C_i \Omega_i, \tag{20}$$

and in the 'rapid precession' mode it is

$$\dot{\psi}_i = -\sigma_i - \Gamma\cos^2\theta_i / C_i\Omega_i. \tag{21}$$

As for the precessional rates given by (17) and (18), from the point of view of the mantle frame, the latter motion is slow and the former rapid.

The spatial angular velocity of the inner core is then easily shown to be

$$\boldsymbol{\omega} = -\hat{\boldsymbol{e}}_3 \Gamma\cos\theta_i/C_i\Omega_i + \hat{\boldsymbol{\epsilon}}_3(\Omega_i + \Gamma\cos^2\theta_i/C_i\Omega_i) \tag{22}$$

in the 'slow precession' mode and

$$\boldsymbol{\omega} = \hat{\boldsymbol{e}}_3(\dot{\phi}_i + \Omega) - \hat{\boldsymbol{\epsilon}}_3(\sigma_i + \Gamma\cos^2\theta_i/C_i\Omega_i) \tag{23}$$

in the 'rapid precession' mode.

Finally, the direction of the angular momentum vector in both modes is indicated by

$$\boldsymbol{L}/A_i = \boldsymbol{\omega} + \hat{\boldsymbol{\epsilon}}_3\sigma_i. \tag{24}$$

Vector diagrams for the two modes showing the connections between the vectors $\boldsymbol{L}/A_i$, $\boldsymbol{\omega}$, $\hat{\boldsymbol{\epsilon}}_3$ and $\hat{\boldsymbol{e}}_3$ are given in figure 6.

Poinsot constructions are easily made for the two modes of regular precession. In 'slow precession', the body cone makes

$$\frac{|\boldsymbol{\omega}\times\hat{\boldsymbol{e}}_3|}{|\boldsymbol{\omega}\times\hat{\boldsymbol{\epsilon}}_3|} = \left(1 + \frac{C_i\Omega_i^2}{\Gamma\cos^2\theta_i}\right)\cos\theta_i = \frac{\dot{\psi}_i\cos\theta_i}{\dot{\psi}_i - \Omega_i} \tag{25}$$

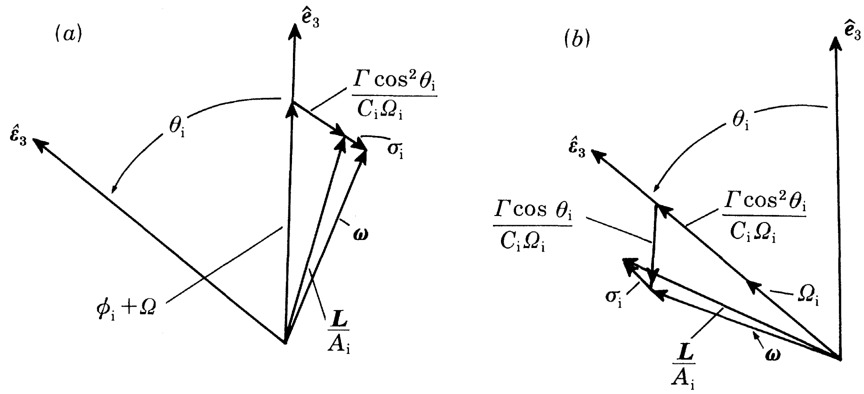

FIGURE 6. Vector diagrams for the two modes of regular precession of the inner core: (*a*) rapid precession; (*b*) slow precession.

revolutions in completing one circuit of the space cone and in 'rapid precession' it makes

$$\frac{|\boldsymbol{\omega}\times\hat{\boldsymbol{\epsilon}}_3|}{|\boldsymbol{\omega}\times\hat{\boldsymbol{e}}_3|} = \left(1+\frac{1}{[(C_i-A_i)/A_i]+[\Gamma\cos^2\theta_i/C_i\Omega_i^2]}\right)\frac{1}{\cos\theta_i} = \frac{\dot{\psi}_i-\Omega_i}{\dot{\psi}_i\cos\theta_i} \tag{26}$$

revolutions. The two motions are illustrated in figure 7.

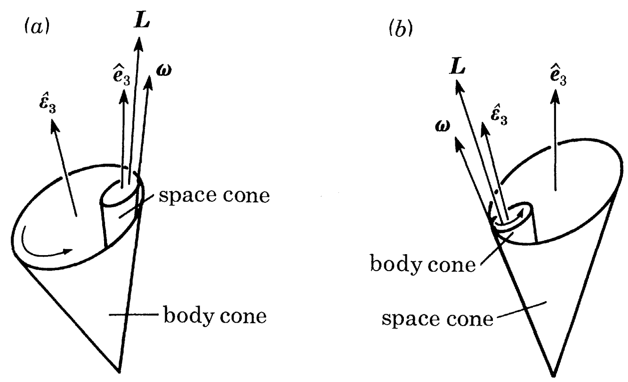

FIGURE 7. Poinsot illustrations of the two modes of regular precession of the inner core: (*a*) rapid precession; (*b*) slow precession.

To discuss the coupled problem of inner core rotation with the compensating motion of the rest of the Earth taken into account, it is necessary to introduce an inertial reference frame $\hat{\boldsymbol{E}}_1$, $\hat{\boldsymbol{E}}_2$, $\hat{\boldsymbol{E}}_3$ and a second set of Euler angles ϕ, θ, ψ which carry this frame into the mantle frame by successive rotations. Figure 8 shows the relationships of the three frames of reference required for the coupled problem.

The spatial angular velocity of the inner core is then written

$$\boldsymbol{\omega} = \dot{\phi}_s\,\hat{\boldsymbol{e}}_3+\dot{\theta}_s\,\hat{\boldsymbol{\epsilon}}_1'+\dot{\psi}_s\,\hat{\boldsymbol{\epsilon}}_3, \tag{27}$$

and that of the mantle is measured from it as

$$\boldsymbol{\omega}-\boldsymbol{\omega}_i = (\dot{\phi}_s-\dot{\phi}_i)\,\hat{\boldsymbol{e}}_3+(\dot{\theta}_s-\dot{\theta}_i)\,\hat{\boldsymbol{\epsilon}}_1'+(\dot{\psi}_s-\dot{\psi}_i)\,\hat{\boldsymbol{\epsilon}}_3. \tag{28}$$

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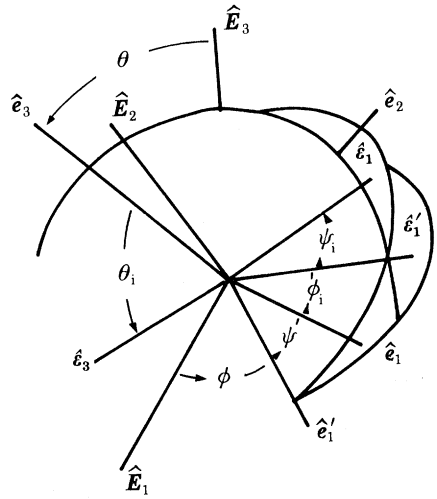

FIGURE 8. Transformation between space $(\hat{\boldsymbol{E}}_1, \hat{\boldsymbol{E}}_2, \hat{\boldsymbol{E}}_3)$, mantle $(\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3)$ and inner core $(\hat{\boldsymbol{\epsilon}}_1, \hat{\boldsymbol{\epsilon}}_2, \hat{\boldsymbol{\epsilon}}_3)$ frames of reference. Space to mantle to inner core transformation is $\phi\hat{\boldsymbol{E}}_3 \rightarrow \theta\hat{\boldsymbol{e}}_1' \rightarrow \psi\hat{\boldsymbol{e}}_3 \rightarrow \phi_i\,\hat{\boldsymbol{e}}_3 \rightarrow \theta_i\,\hat{\boldsymbol{\epsilon}}_1' \rightarrow \psi_i\,\hat{\boldsymbol{\epsilon}}_3$.

There are now three equations of motion for the inner core,

$$\left.\begin{aligned} (\mathrm{d}/\mathrm{d}t)\,(\dot{\theta}_s) + (\sigma_i + \dot{\psi}_i)\,\dot{\phi}_s \sin\theta_i &= -(\Gamma/A_i)\cos\theta_i \sin\theta_i, \\ (\mathrm{d}/\mathrm{d}t)\,(\dot{\phi}_s \sin\theta_i) - (\sigma_i + \dot{\psi}_i)\,\dot{\theta}_s &= 0, \\ \dot{\phi}_s \cos\theta_i + \dot{\psi}_s &= \Omega_i. \end{aligned}\right\} \tag{29}$$

and three for the mantle,

$$\left.\begin{aligned} (\mathrm{d}/\mathrm{d}t)\,(\dot{\theta}_s - \dot{\theta}_i) - (\sigma - \dot{\phi}_i)\,(\dot{\psi}_s - \dot{\psi}_i)\sin\theta_i &= (\Gamma/A)\cos\theta_i \sin\theta_i, \\ (\mathrm{d}/\mathrm{d}t)\,((\dot{\psi}_s - \dot{\psi}_i)\sin\theta_i) + (\sigma - \dot{\phi}_i)\,(\dot{\theta}_s - \dot{\theta}_i) &= 0, \\ (\dot{\phi}_s - \dot{\phi}_i) + (\dot{\psi}_s - \dot{\psi}_i)\cos\theta_i &= \Omega, \end{aligned}\right\} \tag{30}$$

where $\sigma = [(C-A)/A]\,\Omega$ is the Euler angular frequency for the part of the Earth lying outside the inner core. The spatial angular velocities introduced in (27) are related to the Euler angles and their time derivatives by

$$\left.\begin{aligned} \dot{\phi}_s &= \dot{\phi}_i + \dot{\phi}\sin\theta\cot\theta_i\cos(\theta_i + \psi) - \dot{\theta}\cot\theta_i \sin(\phi_i + \psi) + \dot{\phi}\cos\theta + \dot{\psi}, \\ \dot{\theta}_s &= \dot{\theta}_i + \dot{\phi}\sin\theta\sin(\phi_i + \psi) + \dot{\theta}\cos(\phi_i + \psi), \\ \dot{\psi}_s &= \dot{\psi}_i - \dot{\phi}(\sin\theta/\sin\theta_i)\cos(\phi_i + \psi) + (\dot{\theta}/\sin\theta_i)\sin(\phi_i + \psi). \end{aligned}\right\} \tag{31}$$

The formulation given here appears to be very convenient because the six angular velocities $\dot{\phi}_s$, $\dot{\theta}_s$, $\dot{\psi}_s$, $\dot{\phi}_i$, $\dot{\theta}_i$ and $\dot{\psi}_i$ are shown to depend only on the single angular coordinate θ_i.

Again, the systems (29) and (30) can be solved for a regular precession of the inner core in the mantle frame defined by constancy of $\dot{\phi}_i$, θ_i and $\dot{\psi}_i$ (Szeto & Smylie 1984; Smylie *et al.* 1984). $\dot{\phi}_s$, θ_s, $\dot{\psi}_s$ and $\dot{\phi}$, θ, $\dot{\psi}$ are then also constant in the motion. For application to the precession of the dipole field or 'dipole wobble', both the angle of inclination θ_i and the precessional rate $\dot{\phi}_i$ are prescribed. Then the first of equations (30) provides

$$\dot{\psi}_s - \dot{\psi}_i = -\Gamma\cos\theta_i/A(\sigma - \dot{\phi}_i), \tag{32}$$

and the last gives

$$\dot{\phi}_s = \Omega + \dot{\phi}_i + \Gamma\cos^2\theta_i/A(\sigma - \dot{\phi}_i). \tag{33}$$

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With the total system angular momentum vector along $\hat{\boldsymbol{E}}_3$ (the invariable axis), it is found that

$$\theta = -\arctan\left(\frac{C_\mathrm{i}\,\Omega_\mathrm{i}\sin\theta_\mathrm{i} - A_\mathrm{i}\,\dot{\phi}_\mathrm{s}\cos\theta_\mathrm{i}\sin\theta_\mathrm{i} + A(\dot{\psi}_\mathrm{s}-\dot{\psi}_\mathrm{i})\sin\theta_\mathrm{i}}{C\Omega + C_\mathrm{i}\,\Omega_\mathrm{i}\cos\theta_\mathrm{i} + A_\mathrm{i}\,\dot{\phi}_\mathrm{s}\sin^2\theta_\mathrm{i}}\right), \tag{34}$$

and the system (31) becomes

$$\dot{\phi} = -[(\dot{\psi}_\mathrm{s}-\dot{\psi}_\mathrm{i})\sin\theta_\mathrm{i}]/\sin\theta. \tag{35}$$

$\hat{\boldsymbol{\epsilon}}_3$, $\hat{\boldsymbol{e}}_3$ and $\hat{\boldsymbol{E}}_3$ can be shown to be co-planar in regular precession so that $\phi_\mathrm{i}+\psi = 0$ and $\dot{\psi} = -\dot{\phi}_\mathrm{i}$. Finally, the remaining constants of the motion are

$$\dot{\psi}_\mathrm{i} = -\Gamma\cos\theta_\mathrm{i}/C_\mathrm{i}\,\dot{\phi}_\mathrm{s} - [(C_\mathrm{i}-A_\mathrm{i})/C_\mathrm{i}]\,(\dot{\phi}_\mathrm{s}\cos\theta_\mathrm{i} + (\dot{\psi}_\mathrm{s}-\dot{\psi}_\mathrm{s})) \tag{36}$$

and

$$\Omega_\mathrm{i} = -\Gamma\cos\theta_\mathrm{i}/C_\mathrm{i}\,\dot{\phi}_\mathrm{s} + (A_\mathrm{i}/C_\mathrm{i})\,(\dot{\phi}_\mathrm{s}\cos\theta_\mathrm{i} + (\dot{\psi}_\mathrm{s}-\dot{\psi}_\mathrm{i})). \tag{37}$$

Possible observables in the associated motion of the mantle are its polar motion and sway. The polar motion or movement of the mantle's rotation vector in the mantle frame takes place at the rate

$$\dot{\phi}_\mathrm{i}(\dot{\psi}_\mathrm{s}-\dot{\psi}_\mathrm{i})\sin\theta_\mathrm{i}/\Omega\ \mathrm{rad\ s^{-1}}, \tag{38}$$

while the sway or angular departure of the rotation vector of the mantle from the invariable axis is

$$\arctan\left(\frac{\Omega\sin\theta + (\dot{\psi}_\mathrm{s}-\dot{\psi}_\mathrm{i})\sin\theta_\mathrm{i}\cos\theta}{\Omega\cos\theta - (\dot{\psi}_\mathrm{s}-\dot{\psi}_\mathrm{i})\sin\theta_\mathrm{i}\sin\theta}\right). \tag{38}$$

In the space frame it has the prograde rate $\dot{\phi}$ given by (35). Numerical values for the solution of the coupled regular precession problem are listed in table 4 for assumed values of $\theta_\mathrm{i} = 11°$ and $\dot{\phi}_\mathrm{i} = -0.05°$ per year.

The pressure or inertial couple acting on the inner core has been calculated by Kakuta *et al.* (1975) and by Busse (1970). Equation (29) of Kakuta *et al.* gives it as

$$-\mu_r A_\mathrm{i}\,\Omega\,\mathrm{d}\iota_m/\mathrm{d}t, \tag{40}$$

TABLE 4. NUMERICAL RESULTS FOR THE COUPLED REGULAR PRECESSIONAL MOTION OF THE INNER CORE

(Values indicated are for an angular inclination of the inner core axis with respect to the mantle symmetry axis of $\theta_\mathrm{i} = 11°$ and for a precessional rate of $\dot{\phi}_\mathrm{i} = -0.05°$ per year.)

quantity	value
θ_i	11°
$\dot{\phi}_\mathrm{i}$	−0.05° per year
$\dot{\psi}_\mathrm{i}$	-7.2×10^{-7} rad s^{-1}
Ω_i	350° per day
θ	0.019°
$\dot{\phi}$	360° per day
$\dot{\psi}$	0.05° per year
'polar motion'	0.06″ per year
'sway'	0.06″ per year
$\dot{\phi}_\mathrm{s}$	360.6° per day
$\dot{\psi}_\mathrm{s}$	-8.4×10^{-7} rad s^{-1}
σ_i	1.70×10^{-7} rad s^{-1}
σ	2.39×10^{-7} rad s^{-1}

where $\mu_r = 5\times10^{-5}$ is a dimensionless numerical factor and $\iota'_m = \hat{e}_3 - \hat{E}_3\cdot\hat{e}_3$ is inclined to $\hat{E}_3$ at the angle θ and it rotates about $\hat{E}_3$ at the rate $\dot{\phi}$ given by (35).

Thus

$$\left|\frac{\mathrm{d}\iota'_m}{\mathrm{d}t}\right| = |\dot{\phi}|\,|\iota'_m| = \left|\frac{(\dot{\psi}_\mathrm{s}-\dot{\psi}_\mathrm{i})\sin\theta_\mathrm{i}}{\cos\frac{1}{2}\theta}\right|. \tag{41}$$

Substitution from (31) yields the ratio of the inertial torque to gravity torque as

$$\frac{\mu_r A_\mathrm{i}\,\Omega}{A(\sigma-\dot{\phi}_\mathrm{i})\cos\frac{1}{2}\theta} = \mu_r\frac{A_\mathrm{i}}{C-A} = 1.2\times10^{-5}. \tag{42}$$

Since the inertial couple acts in the same direction as the gravity torque, it can be disregarded in inner core dynamics.

From the value in table 4, the angular velocity of the inner core with respect to the mantle is seen to be closely $|\dot{\psi}_\mathrm{i}| = 7.2\times10^{-7}$ rad s^{-1}. On the premise that adjustment to mantle motion is accomplished in a uniform turbulent boundary layer just outside the inner core, the frictional torque (Toomre 1966) amounts to

$$(0.01\text{–}0.1)\times\tfrac{3}{4}\pi^2\rho_\mathrm{i}\,r_0^5|\dot{\psi}_\mathrm{i}| = 1.3\times10^{21}\text{ to }1.3\times10^{22}\text{ N m}. \tag{43}$$

While this torque is again smaller than the gravity torque it is not confined to act along the line of nodes and its influence on inner core dynamics requires further investigation.

The electromagnetic torque on the inner core scales as $10^7 r_0^3 B_\mathrm{p} B_\mathrm{t}$ where B_p and B_t are respectively the scales of the poloidal and toroidal fields at its surface. With poloidal and toroidal fields of like magnitude (10^{-3} T), the torque is of the order 2×10^{19} N m, while for stronger toroidal fields (10^{-2} T), it is 2×10^{20} N m. A much more complex model, however, involving core electrodynamics is required to describe adequately the effect of electromagnetic torques on the rotational dynamics of the inner core.

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